

SPLIT-CM POINTS AND CENTRAL VALUES OF HECKE L-SERIES

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ABSTRACT. Split-CM points are points of the moduli space $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ corresponding to products $E \times E'$ of elliptic curves with the same complex multiplication. We prove that the number of split-CM points in a given class of $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ is related to the coefficients of a weight $3/2$ modular form studied by Eichler. The main application of this result is a formula for the central value $L(\psi_N, 1)$ of a certain Hecke L -series. The Hecke character ψ_N is a twist of the canonical Hecke character ψ for the elliptic \mathbb{Q} -curve A studied by Gross, and formulas for $L(\psi, 1)$ as well as generalizations were proven by Villegas and Zagier. The formulas for $L(\psi_N, 1)$ are easily computable and numerical examples are given.

1. INTRODUCTION

Let $D < 0$, $|D|$ prime be the discriminant of an imaginary quadratic field K with ring of integers \mathcal{O}_K . Suppose N is a prime which splits in \mathcal{O}_K and is divisible by an ideal \mathcal{N} of norm N . We will define Hecke characters ψ_N of K of weight one and conductor \mathcal{N} (see Section 3). These are twists of the canonical Hecke characters studied by Rohrlich [Roh80a, Roh80b, Roh82] and Shimura [Shi64, Shi71, Shi73b]. Denote by $L(\psi_N, s)$ the corresponding Hecke L -series.

Our main theorem (Theorem 3.6) is a formula in the spirit of Waldspurger's results [Wal80, Wal81]. It says approximately that

$$(1.1) \quad L(\psi_N, 1) = \sum_{[R]} \sum_{[\mathfrak{a}]} \Theta_{[\mathfrak{a}, R], \mathcal{N}} \cdot h_{[\mathfrak{a}, R]}^\varepsilon(-N).$$

Here the first sum is over all conjugacy classes of maximal orders R in the quaternion algebra ramified only at ∞ and $|D|$, and the second sum is over the elements $[\mathfrak{a}]$ of the ideal class group of \mathcal{O}_K . We will see that the $h_{[\mathfrak{a}, R]}^\varepsilon(-N)$ are integers related to coefficients of a certain weight $3/2$ modular form, and that the $\Theta_{[\mathfrak{a}, R], \mathcal{N}}$ are algebraic integers equal to the value of a symplectic theta function on 'split-CM' points (defined in Section 3) in the Siegel space $\mathfrak{h}_2/Sp_4(\mathbb{Z})$. We expect the formula (1.1) to be useful for computing the central value $L(\psi_N, 1)$.

Let $A(|D|)$ denote a \mathbb{Q} -curve as defined in [Gro80]. This is an elliptic curve defined over the Hilbert class field H of K with complex multiplication by \mathcal{O}_K which is isogenous over H to its Galois conjugates. Its L -series is a product of the squares of L -series $L(\psi, s)$ over the $h(D)$ Hecke characters of conductor (\sqrt{D}) . A formula for the central value $L(\psi, 1)$ expressed as a square of linear combinations of certain theta functions was proven by Villegas in [RV91]. Extensions of his result to higher weight Hecke characters were given by Villegas in [RV93] and jointly with Zagier in [RVZ93]. The Hecke character ψ_N is a twist of ψ by a quadratic Dirichlet character of conductor $(\sqrt{D})\mathcal{N}$. Therefore our result (1.1) gives a formula for the central value of the corresponding twist of $A(|D|)$.

Our main theorem can be stated in a particularly nice form when the class number of \mathcal{O}_K is one. Then $[\mathfrak{a}] = [\mathcal{N}] = [\mathcal{O}_K]$ and so in particular $\Theta_{[\mathfrak{a}, R], \mathcal{N}} = \Theta_{[R]}$ and $h_{[\mathfrak{a}, R]}^\varepsilon(-N) = h_R^\varepsilon(-N)$.

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are independent of $[\mathfrak{a}]$ and N . This suggests that formula (1.1) will lead to a generating series for $L(\psi_N, 1)$ as N varies in terms of linear combinations (with scalars in $\{\Theta_{[R]}\}$) of half-integer weight modular forms.

We hope to extend these results to higher weight as follows. For certain $k \in \mathbb{Z}_{\geq 1}$ it is well-known that the central value $L(\psi_N^k, k)$ can be written as a trace over the class group of \mathcal{O}_K of a weight k Eisenstein series evaluated at Heegner points of level N and discriminant D . It is a general philosophy (see [Zag02], for example) that such traces relate to coefficients of a corresponding modular form of half-integer weight. By the Siegel-Weil formula¹ we can write the central value of $L(\psi_N^k, s)$ in terms of a sum of theta-series²

$$(1.2) \quad L(\psi_N^k, k) = \sum_{[\mathfrak{a}]} \sum_{[Q]} \frac{1}{\omega_Q} \Theta_Q(\tau_{\mathfrak{a}}).$$

Here the sum is over $[\mathfrak{a}]$ in the class group of \mathcal{O}_K and over classes of positive definite quadratic forms $Q : \mathbb{Z}^{2k} \rightarrow \mathbb{Z}$ in $2k$ variables and in a given genus. The point $\tau_{\mathfrak{a}} \in \mathfrak{h}$ is a Heegner point of level N and discriminant D . Analogous to the case of two variables, these quadratic forms correspond to higher rank Hermitian forms (see [Otr71] and [HK86, HK89, HI80, HI81, HI83]). An approach to counting the number of distinct theta values in (1.2) would be to associate the Hermitian forms to isomorphism classes of rank k R -modules of B , for maximal orders R of B . This paper does this for the case $k = 1$. Our intention here is to lay the groundwork for the generalization to arbitrary weight k .

This paper is organized as follows. Basic notation is given in Section 2. Background and a statement of results are in Section 3. In Section 4, we analyze the endomorphisms of the principally polarized abelian varieties for the split-CM points, and show they form an explicit maximal order in the quaternion algebra B . In Section 5 we identify these orders with explicit right orders in B . In Section 6 we prove the main results (Theorems 3.2, 3.3 and 3.6) and provide numerical examples.

2. NOTATION

Given any imaginary quadratic field M of discriminant $d < 0$, we denote by \mathcal{O}_M its ring of integers, $Cl(\mathcal{O}_M)$ its ideal class group, $h(d)$ its class number, and $Cl(d)$ the isomorphic class group of primitive positive definite binary quadratic forms of discriminant d . A nonzero integral ideal of \mathcal{O}_M with no rational integral divisors besides ± 1 is said to be *primitive*. Any primitive ideal \mathfrak{a} of \mathcal{O}_M can be written uniquely as the \mathbb{Z} -module

$$\mathfrak{a} = a\mathbb{Z} + \frac{-b + \sqrt{d}}{2} \mathbb{Z} = [a, \frac{-b + \sqrt{d}}{2}]$$

with $a := N\mathfrak{a}$ the norm of \mathfrak{a} , and b an integer defined modulo $2a$ which satisfies $b^2 \equiv d \pmod{4a}$. Conversely any $a, b \in \mathbb{Z}$ which satisfy the conditions above determine a primitive ideal of \mathcal{O}_M . The coefficients of the corresponding primitive positive definite binary quadratic form are given by $[a, b, c := \frac{b^2 - D}{4a}]$. The form $[a, -b, c]$ corresponds to the ideal $\bar{\mathfrak{a}}$. We will always assume our forms are primitive positive definite and the same for ideals. The point

$$\tau_{\mathfrak{a}} := \frac{-b + \sqrt{d}}{2a}$$

¹The precise statement of this formula is simplified here for the sake of exposition.

²Here ω_Q is the number of automorphisms of the form Q .

is in the upper half-plane \mathfrak{h} of \mathbb{C} and is referred to in general as a *CM point*. A *Heegner point* of level N and discriminant D is a CM point $\tau_{\mathfrak{a}}$ where \mathfrak{a} is given by a form $[a, b, c]$ of discriminant D such that $N|a$. The *root* of $\tau_{\mathfrak{a}}$ is defined to be the reduced representative $r \in (\mathbb{Z}/2N\mathbb{Z})^\times$ such that $b \equiv r \pmod{2N}$.

Square brackets $[\cdot]$ around an object will denote its respective equivalence class. The units of a ring R are written as R^\times .

3. STATEMENT OF RESULTS

We first recall some basic results for Siegel space and symplectic modular forms.

Assume K is an imaginary quadratic field of prime discriminant $D < -4$. Let L be an imaginary quadratic field of discriminant $-N < 0$ where N is a prime which splits in \mathcal{O}_K , and is divisible by an ideal \mathcal{N} of norm N . Note $h(D)$ and $h(-N)$ are both odd since $|D|$ and N are prime. Let $\mu : \mathcal{O}_K/\mathcal{N} \rightarrow \mathbb{Z}/N\mathbb{Z}$ be the natural isomorphism. Composing this with the Jacobi symbol $(\frac{\cdot}{N}) : \mathbb{Z}/N\mathbb{Z} \rightarrow \{0, \pm 1\}$ defines a character

$$\chi : (\mathcal{O}_K/\mathcal{N})^\times \rightarrow \{\pm 1\}.$$

This is an odd quadratic Dirichlet character of conductor \mathcal{N} . Let $I_{\mathcal{N}}$ denote the group of nonzero fractional ideals of K which are coprime to \mathcal{N} , and let $P_{\mathcal{N}} \subset I_{\mathcal{N}}$ be the subgroup of principal ideals. The map $\psi_{\mathcal{N}} : P_{\mathcal{N}} \rightarrow K^\times$ defined by

$$\psi_{\mathcal{N}}((\alpha)) := \chi(\alpha)\alpha$$

is a homomorphism. There are exactly $h(D)$ extensions of $\psi_{\mathcal{N}}$ to a Hecke character $\psi_{\mathcal{N}} : I_{\mathcal{N}} \rightarrow \mathbb{C}^\times$. This produces $h(D)$ primitive Hecke characters of weight one and conductor \mathcal{N} . (See [Gro84, Pac05] and [Roh80a, p.225] for more details). Fix a choice of $\psi_{\mathcal{N}}$. We can extend $\psi_{\mathcal{N}}$ to a multiplicative function on all of \mathcal{O}_K by setting $\psi_{\mathcal{N}}(\mathfrak{a}) := 0$ if \mathfrak{a} is not coprime to \mathcal{N} .

To $\psi_{\mathcal{N}}$ we associate the Hecke L -function

$$L(\psi_{\mathcal{N}}, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\psi_{\mathcal{N}}(\mathfrak{a})}{N\mathfrak{a}^s}, \quad \operatorname{Re}(s) > 3/2.$$

We now recall a result due to Hecke which gives the central value $L(\psi_{\mathcal{N}}, 1)$ as a linear combination of certain theta series evaluated at CM points. For each primitive ideal \mathcal{Q} of \mathcal{O}_L , the associated theta series is defined by

$$\Theta_{\mathcal{Q}}(\tau) := \sum_{\lambda \in \mathcal{Q}} q^{\mathbf{N}(\lambda)/\mathbf{N}(\mathcal{Q})}, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathfrak{h}.$$

It is a modular form on $\Gamma_0(N)$ of weight one and character $\operatorname{sgn}(\cdot) \left(\frac{-N}{|\cdot|} \right)$ (see [Eic66, p.49], for example).

For each primitive ideal \mathfrak{a} of \mathcal{O}_K with norm prime to N , the product ideal $\mathfrak{a}\bar{\mathcal{N}}$ is of the form $[a_1 N, \frac{-b_1 + \sqrt{D}}{2}]$ for some $a_1, b_1 \in \mathbb{Z}$. The point

$$\tau_{\mathfrak{a}\bar{\mathcal{N}}} := \frac{-b_1 + \sqrt{D}}{2a_1 N} \in \mathfrak{h}$$

is a Heegner point of level N and discriminant D . We will write $\tau_{\mathfrak{a}}$ or just τ for $\tau_{\mathfrak{a}\bar{\mathcal{N}}}$ when the context is clear. Note that as \mathfrak{a} runs over a distinct set of representatives of $Cl(\mathcal{O}_K)$, so does $\mathfrak{a}\bar{\mathcal{N}}$. (The fact that representatives of $Cl(\mathcal{O}_K)$ can be chosen with norm prime to N is in [Cox89, Lemmas 2.3, 2.25], for example.) By \mathfrak{a} we will always mean a primitive ideal with norm prime to \mathcal{N} as above.

Hecke's formula [Hec59] for the central value of $L(\psi_N, s)$ states

$$(3.1) \quad L(\psi_N, 1) = \frac{2\pi}{\omega_N \sqrt{N}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \sum_{[\mathcal{Q}] \in Cl(\mathcal{O}_L)} \frac{\Theta_{\mathcal{Q}}(\tau_{\mathfrak{a}\tilde{N}})}{\psi_{\tilde{N}}(\bar{\mathfrak{a}})}$$

where ω_N is the number of units in \mathcal{O}_L .

The theta function for \mathcal{Q} arises from a certain specialization of a symplectic theta function. Let $Sp_4(\mathbb{Z})$ denote the Siegel modular group of degree 2. Let Γ_θ be the subgroup of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp_4(\mathbb{Z})$ ($\alpha, \beta, \gamma, \delta \in \text{Mat}_2(\mathbb{Z})$) such that both $\alpha^T \gamma$ and $\beta^T \delta$ have even diagonal entries. The group Γ_θ inherits the action of $Sp_4(\mathbb{Z})$ on the Siegel upper half plane $\mathfrak{h}_2 := \{z \in \text{Mat}_2(\mathbb{C}) : {}^T z = z, \text{Im}(z) > 0\}$. Define the symplectic theta function by

$$\theta(z) := \sum_{\vec{x} \in \mathbb{Z}^2} \exp[\pi i {}^T \vec{x} z \vec{x}], \quad z \in \mathfrak{h}_2.$$

The function θ satisfies the functional equation

$$(3.2) \quad \theta(M \circ z) = \chi(M) [\det(\gamma z + \delta)]^{1/2} \theta(z), \quad M \in \Gamma_\theta$$

where $\chi(M)$ is an eighth root of unity which depends on the chosen square root of $\det(\gamma z + \delta)$ but is otherwise independent of z . It is a symplectic modular form on Γ_θ of dimension $-1/2$ with multiplier system χ (see [Eic66, p.43] or [Mum07, p.189], for example)³.

Given a primitive ideal \mathcal{Q} of \mathcal{O}_L , let $Q := [a, b, c]$ represent the corresponding binary quadratic form of discriminant $-N$. The product of the matrix of Q with any Heegner point $\tau_{\mathfrak{a}}$ is the Siegel point

$$Q\tau_{\mathfrak{a}} := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \cdot \tau_{\mathfrak{a}} \in \mathfrak{h}_2.$$

We will refer to points constructed in this way as *split-CM points* of level N and discriminant D . This yields the relation

$$(3.3) \quad \Theta_{\mathcal{Q}}(\tau_{\mathfrak{a}}) = \theta(Q\tau_{\mathfrak{a}})$$

which can be substituted into formula (3.1) to get

$$(3.4) \quad L(\psi_N, 1) = \frac{2\pi}{\omega_N \sqrt{N}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \sum_{[Q] \in Cl(-N)} \frac{\theta(Q\tau_{\mathfrak{a}})}{\psi_{\tilde{N}}(\bar{\mathfrak{a}})}.$$

If $Q \sim Q'$ in $Cl(-N)$, then $Q\tau_{\mathfrak{a}} \sim Q'\tau_{\mathfrak{a}}$ in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$, and if $\mathfrak{a} \sim \mathfrak{a}'$ in $Cl(\mathcal{O}_K)$, then $Q\tau_{\mathfrak{a}} \sim Q\tau_{\mathfrak{a}'}$ in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ (see Remark 6.2 and Lemma 6.12). In addition it is shown in [Pac05, Lemma 53] that these equivalences of Siegel points sustain modulo Γ_θ . The function $\theta/\psi_{\tilde{N}}$ is invariant on such points:

Lemma 3.1. *Fix an ideal $\mathfrak{a} \subset \mathcal{O}_K$ and a prime ideal $\mathcal{N} \subset \mathcal{O}_K$ of norm N . Let Q be a binary quadratic form of discriminant $-N$. Then the value*

$$(3.5) \quad \frac{\theta(Q\tau_{\mathfrak{a}\tilde{N}})}{\psi_{\tilde{N}}(\bar{\mathfrak{a}})}$$

depends only on the class $[Q] \in Cl(\mathcal{O}_L)$ and the class $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$.

Proof. The value $\theta(Q\tau_{\mathfrak{a}\tilde{N}})$ is independent of the class representative of $[Q]$ because equivalent forms represent the same values. That (3.5) is independent of the representative of $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$ is a short calculation using the functional equation for θ in (3.2) and is done in [Pac05, Proposition 22]. \square

³The symplectic theta function is sometimes defined with extra parameters, $\theta(z, u, v)$ where $u, v \in \mathbb{C}^2$, in which case the theta function above is equal to $\theta(z, \vec{0}, \vec{0})$.

Therefore the set of points $[Q]\tau_{[\mathfrak{a}]\tilde{N}}$ as $[Q]$ runs over $Cl(-N)$ and $[\mathfrak{a}]$ runs over $Cl(\mathcal{O}_K)$ are equivalent in $\mathfrak{h}_2/\Gamma_\theta$ and are identified under $\theta/\psi_{\tilde{N}}$. We refer to $[Q]\tau_{[\mathfrak{a}]\tilde{N}}$ as a *split-CM orbit*. Thus to determine which values $\theta(Q\tau_{\mathfrak{a}})$ are equal in (3.4) it is necessary to determine which split-CM orbits $[Q]\tau_{[\mathfrak{a}]\tilde{N}}$ are equivalent modulo Γ_θ . Since $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ is a moduli space for the principally polarized abelian varieties of dimension two ([Mum07] or [BL04, Chp. 8]), the classes of split-CM points are determined by the isomorphism classes of the corresponding varieties.

To describe these, we will recall some basic facts about quaternion algebras. Let $B := (-1, D)_{\mathbb{Q}}$ be the quaternion algebra over \mathbb{Q} ramified at ∞ and $|D|$. Recall two maximal orders R, R' in B are *equivalent* if there exists $x \in B^\times$ such that $R' = x^{-1}Rx$. Moreover, two optimal embeddings $\phi : \mathcal{O}_L \hookrightarrow R$ and $\phi' : \mathcal{O}_L \hookrightarrow R'$ are *equivalent* if there exists $x \in B^\times$ and $r \in R'^\times$ such that $R' = x^{-1}Rx$ and $\phi' = (xr)^{-1}\phi(xr)$. Let \mathcal{R} denote the set of conjugacy classes of maximal orders in B and let $\Phi_{\mathcal{R}}$ denote the set of classes of optimal embeddings of \mathcal{O}_L into the maximal orders of B . Let $\mathcal{R}_N \subset \mathcal{R}$ denote the maximal order classes which admit an optimal embedding of \mathcal{O}_L . Given an optimal embedding $(\phi : \mathcal{O}_L \hookrightarrow R) \in \Phi_{\mathcal{R}}$, let $(\bar{\phi} : \mathcal{O}_L \hookrightarrow R) \in \Phi_{\mathcal{R}}$ denote its quaternionic conjugate, so that $\phi(\sqrt{-N}) = \bar{\phi}(-\sqrt{-N})$. The quotient $\Phi_{\mathcal{R}}/-$ will denote the set $\Phi_{\mathcal{R}}$ modulo this conjugation. Let $h_R(-N)$ denote the number of optimal embeddings of \mathcal{O}_L into R modulo conjugation by R^\times . This number is an invariant of the choice of representative of $[R]$ in \mathcal{R} .

Our first theorem says that the classes of split-CM points in Siegel space correspond to classes of maximal orders in B .

Theorem 3.2. *Fix $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$, $\mathcal{N} \subset \mathcal{O}_K$ a prime ideal of norm N , and $\tau := \tau_{\mathfrak{a}\tilde{N}}$. There is a bijection*

$$\Upsilon_1 : \{Q\tau : [Q] \in Cl(-N)\} / Sp_4(\mathbb{Z}) \longrightarrow \mathcal{R}_N.$$

This map is independent of the choice of representative \mathfrak{a} of $[\mathfrak{a}]$.

Let $\Upsilon_1^{-1}([R])$ for $[R] \in \mathcal{R}_N$ denote the pre-image class in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ and set $\Upsilon_1^{-1}([R]) := \emptyset$ if $[R] \in \mathcal{R} \setminus \mathcal{R}_N$. Our second theorem gives the number of split-CM orbits in a given class.

Theorem 3.3. *Assume the hypotheses of Theorem 3.2. For any $[R] \in \mathcal{R}$,*

$$\# \{[Q]\tau \in \Upsilon_1^{-1}([R]) : [Q] \in Cl(-N)\} = h_R(-N)/2.$$

That is, the number of split-CM orbits in the class in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ corresponding to $[R]$ under Theorem 3.2 is $h_R(-N)/2$.

For a maximal order R of B , define $S_R := \mathbb{Z} + 2R$ and $S_R^0 \subset S_R$ to be the suborder of trace zero elements. The suborder S_R^0 is a rank 3 \mathbb{Z} -submodule of R . Define g_R to be its theta series

$$\begin{aligned} g_R(\tau) &:= \frac{1}{2} \sum_{x \in S_R^0} q^{\mathbb{N}(x)} \\ &= \frac{1}{2} + \sum_{N > 0} a_R(N) q^N, \end{aligned}$$

where $a_R(N)$ are defined by its q -expansion. It is well known that g_R is a weight $3/2$ modular form on $\Gamma_0(4|D|)$. Applying [Gro87, Proposition 12.9] to fundamental $-N$ gives

$$a_R(N) = \frac{\omega_R}{\omega_N} h_R(-N)$$

where ω_R is the cardinality of the set $R^\times / \langle \pm 1 \rangle$.

This gives immediately the following Corollary to Theorem 3.3.

Corollary 3.4. *Assume the hypotheses of Theorem 3.3. For any $[R] \in \mathcal{R}$,*

$$\# \{ [Q]\tau \in \Upsilon_1^{-1}([R]) : [Q] \in Cl(-N) \} = a_R(N) \cdot \frac{2\omega_N}{\omega_R}.$$

That is, the number of split-CM orbits in the class in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ corresponding to $[R]$ under Theorem 3.2 is proportional to the N -th Fourier coefficient of the weight $3/2$ modular form g_R .

The application of Theorems 3.2 and 3.3 to a formula for $L(\psi_N, 1)$ proceeds as follows. Define the following normalization of θ given by [Pac05]:

$$(3.6) \quad \hat{\theta}(Q\tau_{\mathfrak{a}\bar{N}}) := \frac{\theta(Q\tau_{\mathfrak{a}\bar{N}})}{\eta(\bar{N})\eta(\mathcal{O}_K)}$$

where $\eta(z) := e_{24}(z) \prod_{n=1}^{\infty} (1 - e^{2\pi iz})$ for $\text{Im}(z) > 0$ is Dedekind's eta function and the evaluation of η on ideals is defined in Section 6. It is proven in [Pac05, Proposition 23] (see also [HV97]) that the numbers in $\hat{\theta}(Q\tau_{\mathfrak{a}\bar{N}})/\psi_N(\bar{\mathfrak{a}})$ are algebraic integers.

Define

$$\Theta_{[\mathfrak{a}, Q], N} := \frac{\hat{\theta}(Q\tau_{\mathfrak{a}\bar{N}})}{\psi_N(\bar{\mathfrak{a}})}.$$

This is well-defined by Lemma 3.1. The following lemma says that the theta-values which correspond to a given class $[R] \in \mathcal{R}$ under Theorem 3.2 are all equal up to ± 1 .

Lemma 3.5. *Fix $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$, $N \subset \mathcal{O}_K$ a prime ideal of norm N , and $\tau := \tau_{\mathfrak{a}\bar{N}}$. Let $[R] \in \mathcal{R}$. Then the values*

$$(3.7) \quad \{ \Theta_{[\mathfrak{a}, Q], N} : [Q]\tau \in \Upsilon_1^{-1}([R]) \}$$

differ by ± 1 .

Assume Lemma 3.5 holds (see Section 5 for the proof). Given $[R] \in \mathcal{R}_N$ and any $[Q]\tau \in \Upsilon_1^{-1}([R])$, define $\Theta_{[\mathfrak{a}, R], N}$ to be either $\Theta_{[\mathfrak{a}, Q], N}$ or $-\Theta_{[\mathfrak{a}, Q], N}$ so that it satisfies $\text{Re}(\Theta_{[\mathfrak{a}, R], N}) > 0$. Set $\Theta_{[\mathfrak{a}, R], N} := 0$ if $[R] \in \mathcal{R} \setminus \mathcal{R}_N$.

We record the mysterious ± 1 signs appearing in Lemma 3.5 by defining

$$(3.8) \quad \begin{aligned} \varepsilon_{[\mathfrak{a}, R]} : \{ [Q]\tau \in \Upsilon_1^{-1}([R]) \} &\longrightarrow \{ \pm 1 \} \\ [Q]\tau &\mapsto \text{sgn}(\text{Re}(\Theta_{[\mathfrak{a}, Q], N})). \end{aligned}$$

Note $\Theta_{[\mathfrak{a}, Q], N} = \pm \Theta_{[\mathfrak{a}, R], N}$ by construction. This definition assigns, albeit somewhat arbitrarily, a fixed choice of sign for the theta-values as $[Q]$ varies.

We then define a corresponding twisted variant of $h_R(-N)$ by

$$(3.9) \quad h_{[\mathfrak{a}, R]}^{\varepsilon}(-N) := \sum_{[Q]\tau \in \Upsilon_1^{-1}([R])} \varepsilon_{[\mathfrak{a}, R]}([Q]\tau).$$

The formula for $L(\psi_N, 1)$ can now be stated as follows.

Theorem 3.6. *Let $N \subset \mathcal{O}_K$ be a prime ideal of norm N . Then*

$$(3.10) \quad L(\psi_N, 1) = \frac{\pi \cdot \eta(\bar{N})\eta(\mathcal{O}_K)}{\omega_N \sqrt{N}} \sum_{[R] \in \mathcal{R}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \Theta_{[\mathfrak{a}, R], N} \cdot h_{[\mathfrak{a}, R]}^{\varepsilon}(-N).$$

where $\Theta_{[\mathfrak{a}, R], N}$ is an algebraic integer and $h_{[\mathfrak{a}, R]}^{\varepsilon}(-N)$ is an integer with $|h_{[\mathfrak{a}, R]}^{\varepsilon}(-N)| \leq h_R(-N)$.

Remark 3.7. The signs in Lemma 3.5 and hence the function $h_{[\mathfrak{a}, R]}^\varepsilon(-N)$ depend on the character χ which appears in the functional equation (6.5) for θ . In particular, the values of χ depend on the entries of the transformation matrices in Γ_θ which takes one Siegel point to an equivalent one. This value is complicated to compute or even define, and is discussed in detail in [AM75, Sta82] and [Eic66, Appendix to Chp 1]. An arithmetic formula for these signs and for $h_{[\mathfrak{a}, R]}^\varepsilon(-N)$ is yet to be determined. But since the $h_{[\mathfrak{a}, R]}^\varepsilon(-N)$ are a weighted count of optimal embeddings, we expect that, like the $h_R(-N)$, they will be related to coefficients of a half-integer weight modular form. This will be treated in a subsequent paper.

Theorem 3.6 gives us an upper bound on $L(\psi_N, 1)$ in terms of the computable modular form coefficients $h_R(-N)$.

Corollary 3.8. *Assume the hypotheses of Theorem 3.6. Then*

$$|L(\psi_N, 1)| \leq \frac{\pi \cdot |\eta(\bar{N})\eta(\mathcal{O}_K)|}{\omega_N \sqrt{N}} \sum_{[R] \in \mathcal{R}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} |\Theta_{[\mathfrak{a}, R], N}| \cdot h_R(-N).$$

If $h(D) = 1$, then (3.10) has a particularly simple form:

Corollary 3.9. *Assume the hypotheses of Theorem 3.6 and suppose $h(D) = 1$. Then $\Theta_{[\mathfrak{a}, R], N} = \Theta_{[R]}$ and $h_{[\mathfrak{a}, R]}^\varepsilon(-N) = h_{[R]}^\varepsilon(-N)$ are independent of \mathfrak{a} and N and*

$$L(\psi_N, 1) = \frac{\pi \cdot |\eta(\mathcal{O}_K)|^2}{\omega_N \sqrt{N}} \sum_{[R] \in \mathcal{R}} \Theta_{[R]} \cdot h_{[R]}^\varepsilon(-N).$$

We conclude this section with a comment regarding varying N . The set

$$\bigcup_N \{[Q]_{\tau_{[\mathfrak{a}], \bar{N}}} : [Q] \in Cl(-N), [\mathfrak{a}] \in Cl(\mathcal{O}_K), N \subset \mathcal{O}_K \text{ of norm } N\},$$

of split-CM orbits over all prime N with $D \equiv \square \pmod{4N}$ partitions into a finite number of Siegel classes in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$. This has a natural explanation from our viewpoint. As a complex torus, $X_{Q\tau}$ is isomorphic to a product $E \times E'$ of two elliptic curves E, E' defined over $\bar{\mathbb{Q}}$ and with complex multiplication by \mathcal{O}_K . (This is the reason the $Q\tau$ are called ‘split-CM’.) It is a general result of [NN81] that there are only finitely many principal polarizations on a given complex abelian variety up to isomorphism. There are also only finitely many isomorphism classes of elliptic curves with CM by \mathcal{O}_K . Together these imply that the number of classes of Siegel points $(X_{Q\tau}, H_{Q\tau})$ for all split-CM points $Q\tau$ of discriminant D must be finite. See [Pac05, Theorem 58] as well for an alternative interpretation.

4. ENDOMORPHISMS OF X_z PRESERVING H_z

In this section we prove that the endomorphisms of the abelian varieties corresponding to split-CM points give maximal orders in the quaternion algebra $B = (-1, D)_\mathbb{Q}$. Let V, V' be complex vector spaces of dimension 2 with lattices $L \subset V, L' \subset V'$. The analytic and rational representations are denoted by $\rho_a : \text{Hom}(X, X') \rightarrow \text{Hom}_\mathbb{C}(V, V')$ and $\rho_r : \text{Hom}(X, X') \rightarrow \text{Hom}_\mathbb{Z}(L, L')$, respectively. Recall the periods matrices $\Pi, \Pi' \in \text{Mat}_{2 \times 4}(\mathbb{C})$ of X, X' commute with ρ_a and ρ_r in the following diagram

$$(4.1) \quad \begin{array}{ccc} \mathbb{Z}^{2g} & \xrightarrow{\Pi} & \mathbb{C}^g \\ \rho_r(f) \downarrow & & \downarrow \rho_a(f) \\ \mathbb{Z}^{2g'} & \xrightarrow{\Pi'} & \mathbb{C}^{g'} \end{array}$$

(see [BL04], for example).

For any Siegel point $z \in \mathfrak{h}_2$, let $\Pi_z := [z, \mathbf{1}_2] \in \text{Mat}_{2 \times 4}(\mathbb{C})$ be its period matrix, $L_z := \Pi_z \mathbb{Z}^4$ be its defining lattice, and $X_z := \mathbb{C}^2 / L_z$ be its corresponding complex torus. The Hermitian form $\mathcal{H}_z : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}$ defined by $\mathcal{H}_z(u, v) := {}^T u \text{Im}(z)^{-1} \bar{v}$ determines a principal polarization on X_z . As a point in the moduli space $\mathfrak{h}_2 / Sp_4(\mathbb{Z})$, z corresponds to the principally polarized abelian variety (X_z, \mathcal{H}_z) . Throughout Sections 4, 5 and 6, fix a representative \mathfrak{a} of $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$, $\mathcal{N} \subset \mathcal{O}_K$ a prime ideal of norm N , $\tau := \tau_{\mathfrak{a}\mathcal{N}} := \frac{-b_1 + \sqrt{D}}{2a_1 N}$, and a split-CM point $z = Q\tau$ of level N and discriminant D where $Q := [a, b, c]$ is of discriminant $-N$. The endomorphisms of (X_z, \mathcal{H}_z) will be our first main object of study.

We define \mathcal{B} to be the \mathbb{Q} -algebra of endomorphisms of X_z which fix \mathcal{H}_z

$$\mathcal{B} := \{ \alpha \in \text{End}_{\mathbb{Q}}(X_z) : \mathcal{H}_z(\alpha u, v) = \mathcal{H}_z(u, \alpha^\iota v) \quad \forall u, v \in \mathbb{C}^2 \};$$

here ι is the canonical involution inherited from $\text{Mat}_2(K)$ as defined in [Shi73a]. In terms of matrices, let $H_z := \text{Im}(z)^{-1}$ denote the matrix of \mathcal{H}_z with respect to the standard basis of \mathbb{C}^2 . Then viewing $\text{End}_{\mathbb{Q}}(X_z) \subseteq \text{Mat}_2(K)$, the set \mathcal{B} is

$$\mathcal{B} = \{ M \in \text{End}_{\mathbb{Q}}(X_z) : {}^T \bar{M} H_z = H_z M^\iota \}.$$

The bar denotes complex conjugation restricted to K . The map ι sends a matrix M to its adjoint, or equivalently sends M to $\text{Tr}(M) \cdot \mathbf{1}_2 - M$.

We define \mathcal{R}_z to be the \mathbb{Z} -submodule of endomorphisms which fix H_z

$$(4.2) \quad \mathcal{R}_z := \{ M \in \text{End}(X_z) : {}^T \bar{M} H_z = H_z M^\iota \}.$$

The first observation is that \mathcal{B} is isomorphic to a rational definite quaternion algebra.

Proposition 4.1. *\mathcal{B} is isomorphic to B as \mathbb{Q} -algebras.*

Remark 4.2. In [Shi73a, Proposition 2.6], Shimura proves \mathcal{B} is a quaternion algebra over \mathbb{Q} in a much more general setting by showing $\mathcal{B} \otimes \mathbb{Q}$ is isomorphic to $\text{Mat}_2(\mathbb{Q})$. Here we give an alternative proof which explicitly gives the primes ramified in \mathcal{B} .

Proof. We will need the following elementary lemma.

Lemma 4.3. *Suppose $Q_1, Q_2 \in \text{Mat}_2(\mathbb{Z})$ with determinant N . Set $H_i := \text{Im}(Q_i \tau)^{-1}$ and*

$$\mathcal{R}_i := \{ M \in \text{End}(X_{Q_i \tau}) : {}^T \bar{M} H_i = H_i M^\iota \}, \quad i = 1, 2.$$

Let $S = \mathbb{Z}$ or \mathbb{Q} and suppose there exists $A \in GL_2(S)$ such that $Q_2 = (\det A)^{-1} A Q_1^T A$. Then the map

$$(4.3) \quad \begin{aligned} \text{End}_S(X_{Q_1 \tau}) &\longrightarrow \text{End}_S(X_{Q_2 \tau}) \\ M &\mapsto A M A^{-1} \end{aligned}$$

and the induced map

$$\mathcal{R}_1 \otimes_{\mathbb{Z}} S \longrightarrow \mathcal{R}_2 \otimes_{\mathbb{Z}} S$$

are S -algebra isomorphisms.

Proof of Lemma. Let $\Pi_i := [Q_i \tau, \mathbf{1}_2]$ be the period matrices for $Q_i \tau$, $i = 1, 2$. Suppose $M \in \text{End}_S(X_{Q_1 \tau})$. By (4.1), this is if and only if $M \Pi_i = \Pi_i P$ for some $P \in \text{Mat}_4(S)$. Set

$$\tilde{A} := \begin{pmatrix} (\det A^{-1})^T A & 0 \\ 0 & A^{-1} \end{pmatrix} \in GL_4(S).$$

Using the identity $A \Pi_1 \tilde{A} = \Pi_2$ gives

$$(A M A^{-1}) \Pi_2 = \Pi_2 (\tilde{A}^{-1} P \tilde{A}).$$

Clearly $\tilde{A}^{-1}P\tilde{A} \in \text{Mat}_4(S)$, hence $AMA^{-1} \in \text{End}_S(X_{Q_{2\tau}})$.

Furthermore the identity $H_1 = (\det A^{-1})^T A H_2 A$ implies ${}^T(\overline{AMA^{-1}})H_2 = H_2(AMA^{-1})^\iota$ by a straightforward calculation. \square

Define matrices

$$(4.4) \quad A := \frac{1}{2a} \begin{pmatrix} 1 & 0 \\ -b & 2a \end{pmatrix} \in GL_2(\mathbb{Q}) \quad \text{and} \quad Q' := \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}.$$

By Lemma 4.3, \mathcal{B} is isomorphic as a \mathbb{Q} -algebra to

$$\mathcal{B}' := \{M \in \text{End}_{\mathbb{Q}}(X_{Q'\tau}) : {}^T\bar{M}H' = H'M^\iota\}$$

where $H' := \text{Im}(Q'\tau)^{-1}$.

We will compute \mathcal{B}' explicitly. Let $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ for any $\tau \in \mathfrak{h}$. Clearly $X_{Q'\tau} \cong E_\tau \times E_{N\tau}$ as complex tori. The endomorphisms of $X_{Q'\tau}$ are characterized as follows.

Lemma 4.4.

$$\text{End}(X_{Q'\tau}) = \begin{pmatrix} \mathcal{O}_K & \mathbb{Z} + \mathbb{Z}\omega/N \\ N\mathbb{Z} + \mathbb{Z}\bar{\omega} & \mathcal{O}_K \end{pmatrix}$$

where $\omega := a_1 N \tau$.

Assuming this for a moment, we have $\text{End}_{\mathbb{Q}}(X_{Q'\tau}) = \text{Mat}_2(K)$, and a quick calculation shows any $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Mat}_2(K)$ satisfies ${}^T\bar{M}H = HM^\iota$ if and only if $\delta = \bar{\alpha}$ and $\gamma = -N\bar{\beta}$. Therefore

$$\mathcal{B}' = \left\{ \begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in K \right\} \subset \text{Mat}_2(K).$$

The elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{D} & 0 \\ 0 & -\sqrt{D} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{D} \\ N\sqrt{D} & 0 \end{pmatrix}$$

form a basis of \mathcal{B}' and clearly give an isomorphism to $(D, -N)_{\mathbb{Q}}$. We claim $B \cong (D, -N)_{\mathbb{Q}}$. This is a general fact: if p, q are primes with $p \equiv q \equiv 3 \pmod{4}$ and $-p$ is a square modulo q , then $(-p, -q)_{\mathbb{Q}}$ is ramified at ∞ and p only, so $(-p, -q)_{\mathbb{Q}} \cong (-1, p)_{\mathbb{Q}}$. Hence $\mathcal{B} \cong \mathcal{B}' \cong B$ as \mathbb{Q} -algebras.

It remains to prove Lemma 4.4.

Proof of Lemma 4.4. For any quadratic surds $\tau, \tau' \in K$,

$$\text{Hom}(E_\tau, E_{\tau'}) = \{\alpha \in K : \alpha(\mathbb{Z} + \mathbb{Z}\tau) \subseteq \mathbb{Z} + \mathbb{Z}\tau'\}.$$

Since $X_{Q'\tau} \cong E_\tau \times E_{N\tau}$, we have

$$\text{End}(X_{Q'\tau}) = \begin{pmatrix} \text{End}(E_\tau) & \text{Hom}(E_{N\tau}, E_\tau) \\ \text{Hom}(E_\tau, E_{N\tau}) & \text{End}(E_{N\tau}) \end{pmatrix}.$$

We compute. $\text{End}(E_{N\tau}) = \mathcal{O}_K$ since $\mathbb{Z} + \mathbb{Z}a_1 N \tau = \mathcal{O}_K$ and $[1, N\tau]$ is a (proper) fractional \mathcal{O}_K -ideal. Similarly $\text{End}(E_\tau) = \mathcal{O}_K$ since $\mathbb{Z} + \mathbb{Z}\tau$ is a fractional \mathcal{O}_K -ideal.

It is straightforward to check $\mathbb{Z} + \mathbb{Z}a_1 \tau \subseteq \text{Hom}(E_{N\tau}, E_\tau)$. On the other hand, $\text{Hom}(E_{N\tau}, E_\tau) \subseteq \mathbb{Z} + \mathbb{Z}\tau$ by definition, and this is proper containment since otherwise $\mathbb{Z} + \mathbb{Z}N\tau$ would preserve $\mathbb{Z} + \mathbb{Z}\tau$ which is impossible since the former contains \mathcal{O}_K . Therefore $\text{Hom}(E_{N\tau}, E_\tau) = \mathbb{Z} + \mathbb{Z}m\tau$ for some integer $m|a_1$ but a quick calculation shows $m = a_1$ else it divides a_1, b_1 and c_1 whose gcd is assumed to be 1.

It remains to show

$$\text{Hom}(E_\tau, E_{N\tau}) = N\mathbb{Z} + \mathbb{Z}\bar{\omega}.$$

First observe the ideal (N) in \mathcal{O}_K is contained in $\text{Hom}(E_\tau, E_{N\tau})$ since

$$N(\mathbb{Z} + \mathbb{Z}a_1N\tau)(\mathbb{Z} + \mathbb{Z}\tau) \subseteq N(\mathbb{Z} + \mathbb{Z}\tau) \subseteq \mathbb{Z} + N\mathbb{Z}\tau.$$

Furthermore (N) splits as $(N) = \mathcal{N} \cdot \bar{\mathcal{N}}$ where $\mathcal{N} = N\mathbb{Z} + \mathbb{Z}\omega$. Therefore

$$\mathcal{N} \cdot \bar{\mathcal{N}} \subseteq \text{Hom}(E_\tau, E_{N\tau}) \subseteq \mathcal{O}_K,$$

where the last containment follows because $\mathbb{Z} + \mathbb{Z}\tau$ is a proper fractional \mathcal{O}_K -ideal which contains $\mathbb{Z} + \mathbb{Z}N\tau$. But since \mathcal{O}_K is Noetherian, there exists a maximal order M such that

$$\mathcal{N} \cdot \bar{\mathcal{N}} \subseteq \text{Hom}(E_\tau, E_{N\tau}) \subseteq M \subseteq \mathcal{O}_K.$$

Therefore either \mathcal{N} or $\bar{\mathcal{N}}$ is in M . Whichever is contained in M is actually equal to M since they are both prime and hence maximal. But $\text{Hom}(E_\tau, E_{N\tau})$ is not contained in \mathcal{N} . For example, $\bar{\omega} \in \text{Hom}(E_\tau, E_{N\tau})$ but not in \mathcal{N} . Thus

$$\text{Hom}(E_\tau, E_{N\tau}) \subseteq \bar{\mathcal{N}}.$$

Finally since the index $[\bar{\mathcal{N}} : (N)] = N$ is prime, either $\text{Hom}(E_\tau, E_{N\tau})$ is equal to \mathcal{N} or $\bar{\mathcal{N}}$, but we already showed the former is impossible, hence it is the latter. \square

This also completes the proof of Proposition 4.1. \square

Lemma 4.5. *\mathcal{R}_z is isomorphic to an order in B as \mathbb{Z} -algebras, and admits an optimal embedding of \mathcal{O}_L .*

Proof. The first part is immediate.

The embedding is given in matrix form by QS where $S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It is straightforward to check that $(QS)^2 = -N$ and $\frac{1+QS}{2} \in \mathcal{R}_z$ using definition (4.2). An embedding is optimal if it does not extend to any larger order in the quotient field, but this is immediate since \mathcal{O}_L is the maximal order in L . (See [Shi73a] for additional discussion of this order.) \square

The next step is to prove the order \mathcal{R}_z is maximal.

Theorem 4.6. *\mathcal{R}_z is a maximal order.*

Proof. It suffices to show the local order $(\mathcal{R}_z)_p$ is maximal for all primes p . We do this with the following two lemmas.

Lemma 4.7. *$(\mathcal{R}_z)_p$ is maximal for all primes $p \neq 2$.*

Proof of Lemma. Define $\mathcal{R}' := \mathcal{B}' \cap \text{End}(Q'\tau)$ with Q' defined in (4.4). From Lemma 4.4 and the definition of \mathcal{B}' above it is clear that \mathcal{R}' is an order given explicitly by

$$(4.5) \quad \mathcal{R}' = \left\{ \begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha \in \mathcal{O}_K, \beta \in \mathbb{Z} + \mathbb{Z}\omega/N \right\}.$$

Its discriminant is D^2 , which can be computed using the basis

$$(4.6) \quad u_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 := \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}, u_4 = \begin{pmatrix} 0 & \omega/N \\ -\bar{\omega} & 0 \end{pmatrix}.$$

Hence \mathcal{R}' is maximal. For $p \nmid a$, the matrix A from (4.4) is in $\text{Mat}_2(\mathbb{Z}_p)$ and so gives an isomorphism $M \mapsto AMA^{-1}$ from $(\mathcal{R}_z)_p \rightarrow \mathcal{R}'_p$. Hence $(\mathcal{R}_z)_p$ is maximal for $p \nmid a$.

There exists a form $\tilde{Q} = \begin{pmatrix} 2\tilde{a} & \tilde{b} \\ \tilde{b} & 2\tilde{c} \end{pmatrix}$ properly equivalent to Q with $\gcd(2a, \tilde{a}) = 1$ (see [Cox89, p. 25,35], for example). Applying Lemma 4.3 to the pair Q and \tilde{Q} gives $\mathcal{R}_z \cong \mathcal{R}_{\tilde{Q}\tau}$. Hence for $p \nmid a$ we can apply the paragraph above to $\mathcal{R}_{\tilde{Q}\tau}$ to conclude $(\mathcal{R}_z)_p$ is maximal. \square

Lemma 4.8. *$(\mathcal{R}_z)_2$ is maximal.*

Proof of Lemma. Note $\gcd(2a, b) = 1$ because N is prime and b is odd. Define $U := \begin{pmatrix} 1 & 0 \\ -2cx-by & 1 \end{pmatrix}$ and $V := \begin{pmatrix} y & -b \\ x & 2a \end{pmatrix}$ where $x, y \in \mathbb{Z}$ such that $2ay + bx = 1$. Then $UQV = Q'$ where Q' was defined in (4.4). Define $\hat{H} := {}^t U^{-1} H U^{-1}$, $\hat{\mathcal{B}} := \left\{ M \in \text{End}_{\mathbb{Q}}(X_{Q'\tau}) : {}^t M \hat{H} = \hat{H} M \right\}$, and $\hat{\mathcal{R}} := \hat{\mathcal{B}} \cap \text{End}(X_{Q'\tau})$. The period matrix $\Pi' := [Q'\tau, \mathbf{1}_2]$ satisfies $\Pi' = U \Pi_z \tilde{V}$ where $\tilde{V} := \begin{pmatrix} V & 0 \\ 0 & U^{-1} \end{pmatrix} \in \text{Mat}_4(\mathbb{Z})$. Hence the map $M \mapsto U M U^{-1}$ from $\mathcal{R}_z \rightarrow \hat{\mathcal{R}}$ is an isomorphism over \mathbb{Z} . Therefore $(\mathcal{R}_z)_p \cong \hat{\mathcal{R}}_p$ for all primes p . We will show $\hat{\mathcal{R}}_2$ is maximal.

By Lemma 4.3 and the isomorphism $\mathcal{B} \cong \mathcal{B}'$, a basis for \mathcal{B} is given by the set $\{A^{-1}u_i A\}$ with A defined in (4.4) and u_i in (4.6). Hence by above the set $\{v_i := U A^{-1} u_i A U^{-1}\}$ gives a basis for $\hat{\mathcal{B}}$ over \mathbb{Q} . Replace v_i with $2av_i$ for $i = 2, 3$ and v_4 by $2aNv_4$. Then explicitly,

$$\begin{aligned} v_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & v_2 &= \begin{pmatrix} 2a\omega & 0 \\ -Nx(b_1 + 2\omega) & 2a\bar{\omega} \end{pmatrix} \\ v_3 &= \begin{pmatrix} 2aNx & 4a^2 \\ -N(Nx^2 + 1) & -2aNx \end{pmatrix} & v_4 &= \begin{pmatrix} 2aNx\omega & 4a^2\omega \\ -N(Nx^2\omega + \bar{\omega}) & -2aNx\omega \end{pmatrix}. \end{aligned}$$

By Lemma 4.4 we see $v_i \in \hat{\mathcal{R}}$, $i = 1, \dots, 4$. To prove $\hat{\mathcal{R}}_2$ is maximal we will use the elements $\{v_i\}$ to construct a basis of $\hat{\mathcal{R}}_2$ whose discriminant is a unit modulo $(\mathbb{Z}_2)^2$.

Associate any matrix $M := (m_{ij} + n_{ij}\omega) \in \text{Mat}_2(\mathbb{Q}(\omega))$ with $m_{ij}, n_{ij} \in \mathbb{Q}$ to the vector

$$\vec{v}_M := {}^t(m_{11}, n_{11}, m_{12}, n_{12}, m_{21}, n_{21}, m_{22}, n_{22}) \in \mathbb{Q}^8.$$

Denote the vector \vec{v}_{v_i} by \vec{v}_i for simplicity. Let $M_{\text{bas}} \in \text{Mat}_{8 \times 4}(\mathbb{Z})$ be the matrix whose i -th column is \vec{v}_i for $i = 1, \dots, 4$. Given $M \in \text{Mat}_2(K)$, $M \in \hat{\mathcal{B}}$ if and only if

$$(4.7) \quad \vec{v}_M = M_{\text{bas}} \cdot \vec{\alpha}_M$$

for some $\vec{\alpha}_M \in \mathbb{Q}^4$. Moreover $M := (m_{ij} + n_{ij}\omega)$ is in $\text{End}(Q'\tau)$ if and only if

$$(4.8) \quad m_{11}, n_{11}, m_{12}, Nn_{12}, \frac{m_{21} - b_1 n_{21}}{N}, n_{21}, m_{22}, n_{22} \in \mathbb{Z}$$

by (4.5). Let $M_{\text{end}} \in \text{Mat}_{8 \times 8}(\mathbb{Q})$ be the matrix which describes the conditions in (4.8) so that $M \in \text{End}(Q'\tau)$ if and only if $M_{\text{end}} \cdot \vec{v}_M \in \mathbb{Z}^8$. Therefore the elements M of $\hat{\mathcal{R}}$ correspond precisely under (4.7) to $\vec{\alpha}_M \in \mathbb{Q}^4$ such that

$$(4.9) \quad M_{\text{end}} \cdot M_{\text{bas}} \cdot \vec{\alpha}_M \in \mathbb{Z}^8.$$

To show the discriminant of $\hat{\mathcal{R}}$ is 1 mod $(\mathbb{Z}_2)^2$ amounts to finding solutions $\vec{\beta} \in \mathbb{Z}^4$ such that $M_{\text{end}} \cdot M_{\text{bas}} \cdot \vec{\beta} \equiv 0 \pmod{4}$. (Then $\vec{\alpha} := \vec{\beta}/4$ satisfies (4.9).) Three linearly independent solutions for $\vec{\alpha}$ are given by the vectors

$$\vec{\alpha}_5 := {}^t(0, 0, 1, 0)/2, \quad \vec{\alpha}_6 := {}^t(0, 1, 0, 1)/2, \quad \text{and} \quad \vec{\alpha}_7 := {}^t(2, 0, 1, 0)/4.$$

Therefore $\vec{v}_i := M_{\text{bas}} \cdot \vec{\alpha}_i$ gives an element in $\hat{\mathcal{R}}$ for $i = 5, 6, 7$. Consider the set $S := \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Observe the relations

$$\vec{v}_5 = \vec{v}_3/2, \quad \vec{v}_7 = (\vec{v}_1 + \vec{v}_5)/2, \quad \text{and} \quad \vec{v}_6 = (\vec{v}_2 + \vec{v}_4)/2.$$

These imply \vec{v}_5 generates \vec{v}_3 , while \vec{v}_1 and \vec{v}_7 generate \vec{v}_5 , and finally \vec{v}_2 and \vec{v}_6 generate \vec{v}_4 . Accordingly, replace \vec{v}_3 and \vec{v}_4 in S with \vec{v}_6 and \vec{v}_7 so that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_6, \vec{v}_7\}$. Now S is a set of linearly independent vectors over \mathbb{Z} and contained in $\hat{\mathcal{R}}$, hence a basis. A computation (using PARI/GP [PAR08]) of the discriminant of $\hat{\mathcal{R}}$ with respect to this basis shows it is $D^2 \cdot N^2 \cdot a^6$. This is a unit modulo $(\mathbb{Z}_2)^2$ since we may assume a is odd. Hence $\hat{\mathcal{R}}_2$ is maximal. \square

This concludes the proof that \mathcal{R}_z is a maximal order. \square

The next step is to prove \mathcal{R}_z is the right order of an explicit ideal in B . We first recall a result of Pacetti which constructs Siegel points from certain ideals of B .

5. SPLIT-CM POINTS AND RIGHT ORDERS IN B

In this section we identify \mathcal{R}_z with an explicit right order in B . Let \mathcal{M} be a maximal order of B such that there exists $u \in \mathcal{M}$ with $u^2 = D$. (Such an order must exist by Eichler's mass formula). Two left \mathcal{M} -ideals I and I' are in the same *class* if there exists $b \in B^\times$ such that $I = I'b$. The number n of left \mathcal{M} -ideal classes is finite and independent of the choice of maximal order \mathcal{M} . Let \mathcal{J} be the set of n left \mathcal{M} -ideal classes, and recall \mathcal{R} is the set of conjugacy classes of maximal orders in B . (Equivalently, \mathcal{R} is the set of conjugacy classes of right orders with respect to \mathcal{M} , taken without repetition.) The cardinality t of \mathcal{R} is less than or equal to n and is called the *type number*.

Recall $B \cong (D, -N)_{\mathbb{Q}}$ and let $1, u, v, uv$ be a basis for B where $u^2 = D$, $v^2 = -N$, and $uv = -vu$. Define the \mathbb{Z} -module

$$(5.1) \quad I_z := \left\langle \left(\frac{b_1 - u}{2a_1N} \right) av, \left(\frac{b_1 - u}{2a_1N} \right) \left(\frac{N + bv}{2} \right), \frac{b - v}{2}, -a \right\rangle_{\mathbb{Z}}.$$

It is proven in [Pac05, p. 369-372] that I_z is a left ideal for a maximal order $\mathcal{M}_{\mathfrak{a}, [\mathcal{N}]}$ which is independent of the class representative of $[\mathcal{N}]$ and of the form Q , and contains the element u . Let R_z denote the right order of I_z . It is maximal because $\mathcal{M}_{\mathfrak{a}, [\mathcal{N}]}$ is maximal.

We will show that the right order R_z has a natural identification with the maximal order \mathcal{R}_z . To do this, we recall a result of [Pac05] which associates ideals of B to Siegel points. Namely, let (I_R, R) be a pair consisting of a left \mathcal{M} -ideal I_R with maximal right order R . Define the 4-dimensional real vector space $V := B \otimes_{\mathbb{Q}} \mathbb{R}$, so that V/I_R is a real torus. The linear map

$$J : V \rightarrow V$$

$$x \mapsto \frac{u}{\sqrt{|D|}} \cdot x$$

induces a complex structure on V . Hence the data $(V/I_R, J)$ determines a 2-dimensional complex torus. Define a map $\mathcal{E}_R : V \times V \rightarrow \mathbb{R}$ by

$$\mathcal{E}_R(x, y) := \text{Tr}(u^{-1}x\bar{y})/\mathbf{N}(I_R),$$

where $\mathbf{N}(I_R)$ is the norm of the ideal I_R and the 'bar' denotes conjugation in B . It is straightforward to check that \mathcal{E}_R is alternating, satisfies $\mathcal{E}_R(Jx, Jy) = \mathcal{E}_R(x, y)$ for all $x, y \in V$, is integral on I_R , and that the form $\mathcal{H}_R : V \times V \rightarrow \mathbb{C}$ defined by

$$(5.2) \quad \mathcal{H}_R(x, y) := \mathcal{E}_R(Jx, y) + i\mathcal{E}_R(x, y), \quad x, y \in V$$

is positive definite (see [Pac05] for details). Thus \mathcal{E}_R is a Riemann form and so there exists a symplectic basis $\{x_1, x_2, y_1, y_2\}$ of I_R with respect to \mathcal{E}_R . The matrix E_R of \mathcal{E}_R with respect to this basis has determinant

$$\det(E_R) = \mathbf{N}(I_R)^{-4} \mathbf{N}(u)^{-2} \text{disc}(I_R),$$

where we have used the fact that $\text{disc}(I_R) = (\det(u_i u_j))_{ij}$ for any basis $\{u_1, \dots, u_4\}$ of I_R . But the fact that R is maximal implies $\text{disc}(I_R) = D^2 \mathbf{N}(I_R)^4$ [Piz80], [Pac05, Proposition 32], hence $\det(E_R) = 1$. This implies \mathcal{E}_R is of type 1, its matrix is $E_R = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}$, and \mathcal{H}_R is a principal positive definite Hermitian form.

The conclusion is that the data (I_R, J, E_R) determines a Siegel point in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$. The action of a $\gamma \in Sp_4(\mathbb{Z})$ on (I_R, J, E_R) is given as a \mathbb{Z} -linear isomorphism $I_R \rightarrow \gamma(I_R)$, which sends $J \rightarrow \gamma^{-1} \circ J \circ \gamma$, and $\mathcal{E}_R \rightarrow \mathcal{E}_R \circ \gamma$.

Left \mathcal{M} -ideals with the same right order class determine equivalent Siegel points under this construction [Pac05, p. 364]. In other words, there is a well-defined map

$$\mathcal{R} \longrightarrow \mathfrak{h}_2/Sp_4(\mathbb{Z}).$$

This can be seen as follows. Let I and I' be two left \mathcal{M} -ideals with the same right order class $[R]$. Assume first that they are equivalent, that is, $I = I'b$ for some $b \in B^\times$. Then multiplication on the right by b determines a \mathbb{Z} -linear isomorphism

$$\begin{aligned} \gamma : I &\longrightarrow I' \\ x &\mapsto x \cdot b. \end{aligned}$$

Furthermore

$$E(\gamma(x), \gamma(y)) = \frac{\text{Tr}(u^{-1}x \cdot b(\overline{y \cdot b}))}{\mathbf{N}(I)} = E(x, y) \cdot \frac{\mathbf{N}(b)}{\mathbf{N}(I)} = E'(x, y),$$

and since J is a multiplication on the left, and b on the right, clearly $\gamma^{-1} \circ J \circ \gamma = J$. Therefore $(I, J, E) \sim (I', J, E')$ for $I \sim I'$. Now suppose I and I' are not equivalent. Then uI has the same left order and right order class as I but is not equivalent to I (see Lemmas 6.7 and 6.9 below). Since there are at most two classes of left \mathcal{M} -ideals with the same right order class, it must be that $uI \sim I' \sim uIu^{-1}$. It is straightforward to check that the map from I to uIu^{-1} via conjugation by u gives $(I, J, E) \sim (uIu^{-1}, J, E)$ and so by the above case, $(I, J, E) \sim (I', J, E')$.

The ideal I_z in (5.1) corresponds to the Siegel point z under this construction. This is left as an exercise in [Pac05] but can be seen as follows. Let $\{x_1, x_2, y_1, y_2\}$ denote the basis, taken in order, of I_z given in (5.1). A straightforward calculation done by Pacetti shows $\{x_1, x_2, y_1, y_2\}$ is symplectic with respect to \mathcal{E} , and of principal type. Then $\{y_1, y_2\}$ is a basis for the complex vector space (V, J) , and the period matrix for the complex torus $(V/I_z, J)$ is the coefficient matrix of the basis of $\{x_1, x_2, y_1, y_2\}$ in terms of $\{y_1, y_2\}$. It suffices to show this period matrix is $\Pi_z := [z, \mathbf{1}_2]$. Thus one needs to verify

$$\begin{aligned} x_1 &= 2a\tilde{\tau}y_1 + b\tilde{\tau}y_2 \\ x_2 &= b\tilde{\tau}y_1 + 2c\tilde{\tau}y_2, \end{aligned}$$

where $\tilde{\tau} := \frac{-b_1 + \sqrt{|D|}J}{2a_1N}$ is given by the complex multiplication J . This is a simple calculation using the relations $D = b_1^2 - 4a_1c_1N$ and $-N = b^2 - 4ac$.

Note this construction determines an isomorphism $\sigma : I_z \longrightarrow L_z$ by

$$x_1 \mapsto \begin{pmatrix} 2a \\ b \end{pmatrix} \tau, \quad x_2 \mapsto \begin{pmatrix} b \\ 2c \end{pmatrix} \tau, \quad y_1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_2 \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which maps $J \mapsto i$. In particular $\mathcal{H}_{R_z}(x, y) = \mathcal{H}_z|_{L_z \times L_z}(\sigma(x), \sigma(y))$ for all $x, y \in I_z$.

The elements of R_z and \mathcal{R}_z can now be related as follows. Any $b \in R_z$ preserves I_z (on the right) as well as the complex structure J and hence defines an endomorphism f_b of X_z . Likewise, any $M \in \mathcal{R}_z$ defines an endomorphism f_M of the torus X_z by definition. We claim these rings give the same endomorphisms of X_z :

Proposition 5.1. *As endomorphisms, R_z is identified with \mathcal{R}_z .*

Proof. Suppose $f_b \in \text{End}(X_z)$ for some $b \in R_z$. To show f_b comes from \mathcal{R}_z , it suffices to show $\rho_r(f_b)$ preserves $H_z|_{L_z \times L_z}$. Equivalently by the map σ it suffices to show

$$\mathcal{H}_{R_z}(x \cdot b, y) = \mathcal{H}_{R_z}(x, y \cdot b').$$

But this is immediate since $\text{Tr}(u^{-1}(xb)\bar{y}) = \text{Tr}(u^{-1}x(\bar{b}\bar{y})) = \text{Tr}(u^{-1}x(\overline{yb}))$ and $\bar{b} = b'$ in B . Therefore as endomorphisms R_z is contained in \mathcal{R}_z . Conversely any $f_M \in \text{End}(X_z)$ for $M \in \mathcal{R}_z$ defines a linear map from I_z to itself which commutes with the complex structure J , hence corresponds to an element in R_z . \square

Corollary 5.2. \mathcal{R}_z is isomorphic to the maximal right order R_z in the quaternion algebra \mathcal{B} , and this map sends $\frac{1+QS}{2} \mapsto \frac{1+v}{2}$.

Proof. The first part follows immediately from the proposition. Regarding the embedding, the rational representation in $\text{Mat}_4(\mathbb{Z})$ of the endomorphism $\frac{1+QS}{2} \in R_z$ is

$$\begin{pmatrix} \frac{b+1}{2} & c & 0 & 0 \\ -a & \frac{1-b}{2} & 0 & 0 \\ 0 & 0 & \frac{1-b}{2} & a \\ 0 & 0 & -c & \frac{b+1}{2} \end{pmatrix}.$$

Its action on the basis x_1, x_2, y_1, y_2 of I_z shows immediately that it is the linear transformation given by multiplication on the right by $\frac{1+v}{2}$. \square

6. FORMULA FOR THE CENTRAL VALUE $L(\psi_N, 1)$

In this section we prove Theorems 3.2, 3.3 and 3.6.

Proof of Theorems 3.2 and 3.3. Fix $[\mathfrak{a}] \in \text{Cl}(\mathcal{O}_K)$, $\mathcal{N} \subset \mathcal{O}_K$ a prime ideal of norm N , $\tau := \tau_{\mathfrak{a}\bar{\mathcal{N}}}$. Throughout the rest of this section, fix $z := Q\tau$ and $z' := Q'\tau$ where Q, Q' are binary quadratic forms of discriminant $-N$. Define

$$(6.1) \quad \Upsilon_1 : \{Q\tau : [Q] \in \text{Cl}(-N)\} / Sp_4(\mathbb{Z}) \longrightarrow \mathcal{R}_N$$

$$[Q]\tau \mapsto [R_{Q\tau}]$$

Given an $R_{Q\tau}$, let $\phi_Q : \mathcal{O}_L \hookrightarrow R_{Q\tau}$ be the optimal embedding defined in Lemma 4.5 and Corollary 5.2. Define a second map

$$(6.2) \quad \Upsilon_2 : \text{Cl}(-N) \longrightarrow \Phi_{\mathcal{R}} / -$$

$$[Q] \mapsto [\phi_Q : \mathcal{O}_L \hookrightarrow R_{Q\tau}].$$

We will start by showing that the maps Υ_1 and Υ_2 are well-defined. First note Υ_1 is injective: if $R_{Q\tau} \sim R_{Q'\tau}$ in B , then we saw in the last section that Pacetti's map $\mathcal{R} \longrightarrow \mathfrak{h}_2/Sp_4(\mathbb{Z})$ sends $R_{Q\tau} \mapsto Q\tau$. After proving the maps are well-defined, we will prove Υ_2 is a bijection and independent of the choice of representative \mathfrak{a} of $[\mathfrak{a}]$. This will simultaneously prove Theorems 3.3 and 3.2.

Lemma 6.1. *If $z \sim z'$ in $\mathfrak{h}_2/\Gamma_\theta$, then $R_z \sim R_{z'}$ in \mathcal{R} .*

Remark 6.2. Note that if $Q \sim Q'$ with $Q = {}^t A Q' A$ for some $A \in SL_2(\mathbb{Z})$, then $Q\tau \sim Q'\tau$ as Siegel points via the matrix $\begin{pmatrix} {}^t A & 0 \\ 0 & A^{-1} \end{pmatrix} \in \Gamma_\theta$.

Proof. Recall $z \sim z'$ in $\mathfrak{h}_2/Sp_4(\mathbb{Z})$ if and only if the abelian varieties (X_z, H_z) and $(X_{z'}, H_{z'})$ are isomorphic. Write X, X', H, H' for $X_z, X_{z'}, H_z, H_{z'}$, respectively. Suppose $f : X \longrightarrow X'$ is

an isomorphism of (X, H) with (X', H') , so that $H'(f(x), f(y)) = H(x, y)$ for all $x, y \in \mathbb{C}^2$. We claim the isomorphism

$$(6.3) \quad \begin{aligned} \text{End}(X) &\longrightarrow \text{End}(X') \\ \alpha &\mapsto f \circ \alpha \circ f^{-1} \end{aligned}$$

induces an isomorphism of \mathcal{R}_z and $\mathcal{R}_{z'}$. This follows immediately from the calculation

$$\begin{aligned} H'(f \circ \alpha \circ f^{-1}(x), y) &= H(\alpha(f^{-1}(x)), f^{-1}(y)) \\ &= H(f^{-1}(x), \alpha'(f^{-1}(y))) && (\text{since } \alpha \in \mathcal{R}_z) \\ &= H'(x, f(\alpha'(f^{-1}(y)))) \\ &= H'(x, (f \circ \alpha \circ f^{-1})^\iota). \end{aligned}$$

The last equality follows because, as a matrix, $\rho_a(f)^\iota = \rho_a(f)^{-1} \det(\rho_a(f))$ and so the determinants in $(f \circ \alpha \circ f^{-1})^\iota$ cancel out. Therefore $\mathcal{R}_{z'} = f \circ \mathcal{R}_z \circ f^{-1}$ and so by Proposition 5.1, $R_z \sim R_{z'}$ in \mathcal{B} . \square

Lemma 6.3. *If $Q \sim Q'$ in $Cl(-N)$, then the corresponding optimal embeddings $\frac{v+1}{2} \hookrightarrow R_z$ and $\frac{v+1}{2} \hookrightarrow R_{z'}$ are equivalent.*

Proof. Suppose $Q \sim Q'$ with $Q' = AQ^T A$ for some $A \in SL_2(\mathbb{Z})$. Then by Lemma 4.3, the map $\mathcal{R}_z \rightarrow \mathcal{R}_{z'}$ by $M \mapsto AMA^{-1}$ is a \mathbb{Z} -algebra isomorphism, and extends to a \mathbb{Q} -algebra isomorphism from $\mathcal{B} \rightarrow \mathcal{B}'$. In particular it sends $QS \mapsto A(QS)A^{-1} = Q'S$. By Corollary 5.2, this induces a \mathbb{Z} -algebra isomorphism of $R_z \rightarrow R_{z'}$ which sends v to v , and extends to a \mathbb{Q} -algebra automorphism of B . Hence by the Skolem-Noether theorem, the map $R_z \rightarrow R_{z'}$ must be conjugation by some unit of B . \square

We now turn to proving Υ_2 is a bijection. The following six lemmas will be needed to prove Υ_2 is injective. Let \mathcal{Q} denote the ideal in L which corresponds to Q .

Lemma 6.4.

$$I_z \cong \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$$

as right \mathcal{O}_L -modules.

Proof of Lemma. Define $v_1 := x_1, v_2 := x_2, v_3 := y_1, v_4 := -y_2$ where x_i, y_j is the basis of I_z defined in Section 4. The $\{v_i\}$ also form a basis for I_z . The map $f : I_z \rightarrow \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$ defined by

$$\begin{aligned} v_1 &\mapsto (a, 0) & v_2 &\mapsto \left(\frac{b - \sqrt{-N}}{2}, 0\right) \\ v_4 &\mapsto (0, a) & v_3 &\mapsto \left(0, \frac{b - \sqrt{-N}}{2}\right) \end{aligned}$$

and extended \mathbb{Z} -linearly is an isomorphism of \mathbb{Z} -modules. To show it is an \mathcal{O}_L -module isomorphism, it suffices to show

$$f\left(v_i\left(\frac{b+v}{2}\right)\right) = f(v_i)\left(\frac{b + \sqrt{-N}}{2}\right) \quad \text{for all } i = 1, 2, 3, 4.$$

For this, use the identities:

$$\begin{aligned} v_1\left(\frac{b+v}{2}\right) &= bv_1 - av_2 & v_3\left(\frac{b+v}{2}\right) &= cv_4 \\ v_2\left(\frac{b+v}{2}\right) &= cv_1 & v_4\left(\frac{b+v}{2}\right) &= -av_3 + bv_4. \end{aligned}$$

□

Lemma 6.5. *Suppose $S := I_z x$ where $x \in B^\times$ commutes with $\frac{v+1}{2}$. Then*

$$S \cong \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}},$$

as right \mathcal{O}_L -modules.

Proof of Lemma. By Lemma 6.4 and the hypotheses on x , the composition from $S \rightarrow \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$ given by $g(v_i x) := f(v_i)$ is an isomorphism of \mathcal{O}_L -modules. □

Lemma 6.6. *Suppose $\bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}} \cong \bar{\mathcal{Q}}' \oplus \bar{\mathcal{Q}}'$ as right \mathcal{O}_L -modules, and $h(-N)$ is odd. Then*

$$Q \sim Q'$$

in $Cl(-N)$.

Proof of Lemma. By a classical theorem of Steinitz [Mil71, Theorem 1.6], $\bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}} \cong \bar{\mathcal{Q}}' \oplus \bar{\mathcal{Q}}'$ as right \mathcal{O}_L -modules if and only if $[\bar{\mathcal{Q}}']^2 = [\bar{\mathcal{Q}}]^2$ as classes in the ideal class group of \mathcal{O}_L . This is if and only if $[\bar{\mathcal{Q}}'/\bar{\mathcal{Q}}]^2 = [\text{id}]$ where id is the identity class. But since the class number $h(-N)$ is odd, this implies $[\bar{\mathcal{Q}}] = [\bar{\mathcal{Q}}']$ in $Cl(-N)$. □

The next three lemmas we need are general results for quaternion algebras. Assume for Lemmas 6.7, 6.8, and 6.9 below that B is a quaternion algebra ramified precisely at ∞ and a prime p . In addition, assume M and R are maximal orders and there exists $u \in M$ such that $u^2 = -p$.

Lemma 6.7.

$$uMu^{-1} = M.$$

Proof. This is clear locally at primes $q \neq p$ because $u^{-1} = -u/p$. This is also clear locally at p because there is a unique maximal order in the division algebra B_p (see [MR03, Theorem 6.4.1, p.208] or [Vig80] for example). □

Lemma 6.8. *Suppose I, I' are left M -ideals with right order R . In addition assume R admits an embedding of a ring of integers \mathcal{O} of some imaginary quadratic field. Set $J := I(I')^{-1}$. Then*

$$JI' \cong I'$$

as right \mathcal{O} -modules.

Proof. First note J is a bilateral M -ideal. Since $u \in M$, $uM = Mu$ by Lemma 6.7 and so is a principal M -ideal of norm p . Hence it is the unique integral bilateral M -ideal of norm p , and so every bilateral M -ideal is equal to $uM \cdot m$ for some $m \in \mathbb{Q}$ [Eic73, Proposition 1, p. 92]. In particular, this implies the bilateral M -ideals are principal. Therefore $J = tM = Mt$ for some $t \in B^\times$, and the map,

$$\begin{aligned} f : I' &\longrightarrow JI' \\ w &\longrightarrow tw \end{aligned}$$

is a \mathbb{Z} -module isomorphism. Since the multiplication by t is on the left, f is an isomorphism of right \mathcal{O} modules. □

Lemma 6.9. *Suppose I is a left M -ideal with right order R . Then uI is also a left M -ideal with right order R . Furthermore, any left M -ideal with right order R is equivalent to I or uI (or both).*

Proof. The right order of uI is clearly R . The left order is $uMu^{-1} = M$ by Lemma 6.7.

Suppose J is any left M -ideal with right order R . The ideal $I^{-1}J$ is R -bilateral, hence

$$I^{-1}J = \mathcal{P}^i m, \quad i = 0, 1, m \in \mathbb{Q}$$

where \mathcal{P} is the unique bilateral R -ideal of norm p [Eic73, Proposition 1, p. 92].

If $I^{-1}J$ is principal, then $I \sim J$. Otherwise $i = 1$. Then since the ideal $I^{-1}uI$ is R -bilateral of norm p , by uniqueness $I^{-1}uI = \mathcal{P}$ and so

$$I^{-1}J = I^{-1}uI \cdot m.$$

Multiplying through by I we see $J \sim uI$ as left M -ideals. □

Now the injectivity of Υ_2 can be proven.

Proposition 6.10. *Suppose $(R_z, \frac{v+1}{2}) \sim (R_{z'}, \frac{v+1}{2})$. Then $Q \sim Q'$ in $Cl(-N)$.*

Proof. The assumption $(R_z, \frac{v+1}{2}) \sim (R_{z'}, \frac{v+1}{2})$ implies there exists $x \in B^\times$ such that

$$x^{-1}R_zx = R_{z'}$$

and $r \in R_{z'}^\times$ such that

$$(xr)^{-1} \left(\frac{v+1}{2} \right) xr = \frac{v+1}{2}.$$

The proof is broken up into two cases.

Case 1. Assume $I_z \sim I_{z'}$. Then $I_zx \sim I_{z'}$ and they both have right order $R_{z'}$. Set $J := I_zxI_{z'}^{-1}$. Then $JI_{z'} \cong I_{z'}$ as right \mathcal{O}_L -modules by Lemma 6.8. Combining with Lemma 6.4 applied to $I_{z'}$ implies

$$JI_{z'} \cong \bar{Q}' \oplus \bar{Q}'$$

as right \mathcal{O}_L -modules.

On the other hand, $JI_{z'} = I_zx$. Since r is a unit, $I_zx = I_zxr$, so replacing x by xr if necessary we may assume $r = 1$ and $x^{-1}(\frac{v+1}{2})x = \frac{v+1}{2}$. Lemma 6.5 applied to I_zx gives

$$JI_{z'} \cong \bar{Q} \oplus \bar{Q}$$

as right \mathcal{O}_L -modules. Hence $Q \sim Q'$ by Lemma 6.6.

Case 2. Assume $I_z \not\sim I_{z'}$. For each maximal order R , there can be at most two left M -ideal classes with right orders in the class $[R]$. Therefore since $I_{z'}$ has right order $R_{z'} \in [R_z]$, but $I_z \not\sim I_{z'}$, by Lemma 6.9 it must be that

$$uI_z \sim I_{z'};$$

note uI_z is a left M -ideal by Lemma 6.7. Then $uI_zx \sim I_{z'}$ and they have the same right order. Let $J := uI_zxI_{z'}^{-1}$ and use the same argument from Case 1, noting that Lemmas 6.4 and 6.5 hold with I_z replaced by uI_z since the multiplication by u is on the left. This concludes the proof that Υ_2 is injective. □

It remains to show that Υ_2 is a surjection. This follows from the fact:

Lemma 6.11.

$$h(-N) = \#\Phi_{\mathcal{R}} / -.$$

Proof of Lemma. For $[R] \in \mathcal{R}$, let $h_R(-N)$ denote the number of optimal embeddings of \mathcal{O}_L into R , modulo conjugation by R^\times . Then

$$\begin{aligned} \#\Phi_{\mathcal{R}}/- &= \frac{1}{2} \sum_{[R] \in \mathcal{R}} h_R(-N) && \text{by definition,} \\ &= h(-N) && \text{by Eichler's mass formula [Gro84, (1.12)].} \end{aligned}$$

□

The last task is to prove the maps Υ_1 and Υ_2 are independent of the choice of representative \mathfrak{a} of $[\mathfrak{a}]$. In fact we will prove a slightly stronger result regarding the right orders:

Lemma 6.12. *If $\mathfrak{a} \sim \mathfrak{a}'$ in $Cl(\mathcal{O}_K)$ then $R_{Q\tau_{\mathfrak{a}\bar{N}}} = R_{Q\tau_{\mathfrak{a}'\bar{N}}}$.*

Proof. The hypothesis $\mathfrak{a} \sim \mathfrak{a}'$ implies $\mathfrak{a}\bar{N} \sim \mathfrak{a}'\bar{N}$. Suppose \bar{N} corresponds to a form $[N, b, c]$. Then we can choose bases so that the products $\mathfrak{a}\bar{N}$, $\mathfrak{a}'\bar{N}$ both correspond to forms with middle coefficient congruent to $b \pmod{2N}$ (see [RV91, Lemma 2.3], for example). The CM-points $\tau_{\mathfrak{a}\bar{N}}, \tau_{\mathfrak{a}'\bar{N}}$ are Heegner points of level N and discriminant D by construction, and by the comment above they have the same ‘root’ $b \pmod{2N}$ of $\sqrt{D} \pmod{4N}$. Hence there exists $M := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N)$ such that

$$M(\tau_{\mathfrak{a}\bar{N}}) = \tau_{\mathfrak{a}'\bar{N}}.$$

Set

$$\tilde{M} := \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$$

where

$$\tilde{\alpha} := \alpha \cdot \mathbf{1}_2, \quad \tilde{\beta} := \beta \cdot Q, \quad \tilde{\gamma} := \gamma \cdot Q^{-1}, \quad \tilde{\delta} := \delta \cdot \mathbf{1}_2.$$

It is shown in [AM75, p.233], for example, that $\tilde{M} \in \Gamma_\theta \subseteq Sp_4(\mathbb{Z})$. Therefore the relation

$$\tilde{M}(Q\tau_{\mathfrak{a}\bar{N}}) = Q\tau_{\mathfrak{a}'\bar{N}}$$

implies $Q\tau_{\mathfrak{a}\bar{N}} \sim Q\tau_{\mathfrak{a}'\bar{N}}$ in $\mathfrak{h}_2/\Gamma_\theta$. Let $\tau := \tau_{\mathfrak{a}\bar{N}}$ and $\tau' := \tau_{\mathfrak{a}'\bar{N}}$. An isomorphism $f_M : X_{Q\tau'} \rightarrow X_{Q\tau}$ is given by

$${}^T(\tilde{\gamma}Q\tau + \tilde{\delta})[Q'\tau, \mathbf{1}_2] = [Q\tau, \mathbf{1}_2] {}^T \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}.$$

The analytic representation of this isomorphism, which we will also denote by f_M , is

$$f_M = {}^T(\tilde{\gamma}Q\tau + \tilde{\delta}) = (\gamma\tau + \delta) \cdot \mathbf{1}_2,$$

where recall $\gamma, \delta \in \mathbb{Z}$. Therefore the map

$$\begin{aligned} \text{End}(X_{Q\tau'}) &\longrightarrow \text{End}(X_{Q\tau}) \\ A &\mapsto f_M A f_M^{-1} = A \end{aligned}$$

is the identity map, hence $\text{End}(X_{Q\tau'}) = \text{End}(X_{Q\tau})$. Moreover the equivalence

$${}^T \bar{A} H_{Q\tau} = H_{Q\tau} A' \quad \Leftrightarrow \quad {}^T \bar{A} Q' = Q' A'$$

implies the relation on the left hand side is independent of τ . Hence $R_{Q\tau} = R_{Q\tau'}$. □

It follows immediately since $R_{Q\tau} = R_{Q\tau'}$ that the maps Υ_1 and Υ_2 are independent of the choice of representative \mathfrak{a} of $[\mathfrak{a}]$.

This completes the proofs of Theorems 3.2 and 3.3. □

Recall the definitions of: the normalized theta values $\Theta_{[\mathfrak{a}, R], N}$ in (3.6), the sign function $\varepsilon_{[\mathfrak{a}, R]}$ on the embeddings in (3.8), and the twisted number of optimal embeddings $h_{[\mathfrak{a}, R]}^\varepsilon(-N)$ in (3.9). The η function in (3.6) is defined on an ideal $\mathfrak{a} = [a, \frac{-b + \sqrt{D}}{2}]$ of \mathcal{O}_K by

$$(6.4) \quad \eta(\mathfrak{a}) := e_{48}(a(b+3)) \cdot \eta\left(\frac{-b + \sqrt{D}}{2a}\right)$$

where $e_n(x) := \exp(2\pi i x/n)$ for $n \in \mathbb{Z}$, $x \in \mathbb{C}$, and $\eta(z) := e_{24}(z) \prod_{n=1}^{\infty} (1 - e^{2\pi i z})$ for $\text{Im}(z) > 0$ is Dedekind's eta function. Using Shimura's reciprocity law it can be shown that the value $\Theta_{[\mathfrak{a}, R], N}$ is an algebraic integer (see [Pac05, Proposition 23, p. 355] and [HV97]).

We now prove Lemma 3.5.

Proof of Lemma 3.5. Theorem 31 of [Pac05] says that if $Q\tau_{\mathfrak{a}\bar{N}} \sim Q'\tau_{\mathfrak{a}\bar{N}}$ in $\mathfrak{h}_2/\Gamma_\theta$, then

$$\Theta_{[\mathfrak{a}, Q], N} = \pm \Theta_{[\mathfrak{a}, Q'], N}.$$

The lemma therefore follows immediately by this fact and Theorem 3.2. \square

We now prove Theorem 3.6.

Proof of Theorem 3.6. The remaining step in deriving formula (3.10) for $L(\psi_N, 1)$ is to determine how θ behaves on equivalent split-CM points. The following is a special case of [Pac05, Theorem 31] but we give a slightly simplified proof.

Lemma 6.13. *Let Q and Q' be binary quadratic forms of discriminant $-N$. If $Q\tau \sim Q'\tau$ in $\mathfrak{h}_2/\Gamma_\theta$, then $\theta(Q\tau) = \pm\theta(Q'\tau)$.*

Proof of Lemma. Suppose $Q\tau \sim Q'\tau$ in $\mathfrak{h}_2/\Gamma_\theta$. Then there exists $M := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_\theta$ such that $M(Q\tau) = Q'\tau$. Recall the functional equation for θ is

$$(6.5) \quad \theta(M \circ z) = \chi(M)[\det(\gamma z + \delta)]^{1/2} \theta(z), \quad M \in \Gamma_\theta$$

where $\chi(M)$ is a certain 8th root of unity.

Then

$$\frac{\theta(Q'\tau)}{\theta(Q\tau)} = \chi(M)[\det(\gamma Q\tau + \delta)]^{1/2}.$$

Applying Smith Normal Form, there exists $U, V \in SL_2(\mathbb{Z})$ such that $UQV = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$, and $U', V' \in SL_2(\mathbb{Z})$ such that $U'Q'V' = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}$. These give isomorphisms $f_U : X_{Q\tau} \rightarrow E_\tau \times E_{N\tau}$ and $f_{U'} : X_{Q'\tau} \rightarrow E_\tau \times E_{N\tau}$ respectively. From the relation $M(Q\tau) = Q'\tau$, we also get an isomorphism $f_M : X_{Q\tau} \rightarrow X_{Q'\tau}$ given by

$${}^T(\gamma Q\tau + \delta)[Q'\tau, \mathbf{1}_2] = [Q\tau, \mathbf{1}_2] {}^T \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Thus the composition

$$f_U \circ f_M \circ f_{U'}^{-1} : E_\tau \times E_{N\tau} \longrightarrow E_\tau \times E_{N\tau}$$

is an automorphism, and the determinant of its analytic representation is a unit and an algebraic integer. This last fact follows from linear algebra or can be deduced directly using Lemma 4.4. Since U and U' are both in $SL_2(\mathbb{Z})$, we get $\det(\gamma Q\tau + \delta) \in \mathcal{O}_K^\times$. Since $D < -4$ this implies $\det(\gamma Q\tau + \delta) = \pm 1$. Therefore $[\det(\gamma Q\tau + \delta)]^{1/2} = \pm\sqrt{\pm 1}$.

This proves $\frac{\theta(Q'\tau)}{\theta(Q\tau)} = \pm\sqrt{\pm 1} \cdot \chi(M)$. But by Theorem 17 of [Pac05], the ratio of theta values on the left is an algebraic integer in the Hilbert class field of K . Hence $\pm\sqrt{\pm 1} \cdot \chi(M)$ is an 8th root of unity and an algebraic integer in the Hilbert class field of K , which does not contain i . Therefore

$$\pm\sqrt{\pm 1} \cdot \chi(M) = \pm 1.$$

□

The theorem follows immediately from Lemma 6.13 and Theorems 3.2 and 3.3. □

7. EXAMPLES

This section provides tables for two class number one examples. All calculations were done in gp/PARI [PAR08]. Given D of class number one, for each admissible N we compute a form $[N, b_1, c_1]$ corresponding to \mathcal{N} . We set $\mathfrak{a}\mathcal{N} = \mathcal{N}$ since $Cl(\mathcal{O}_K)$ is trivial, and $\tau_{\mathfrak{a}\mathcal{N}} := \tau_{\mathcal{N}} := \frac{-b_1 + \sqrt{D}}{2N}$ to be a Heegner point of level N and discriminant D . We choose $[1, \frac{-b_1 + \sqrt{D}}{2}]$ for a basis of \mathcal{O}_K so that following definition (6.4),

$$\eta(\mathcal{N})\eta(\mathcal{O}_K) := e_{48}^2(N(b_1 + 3)^2) \cdot \eta\left(\frac{-b_1 + \sqrt{D}}{2N}\right) \cdot \eta\left(\frac{-b + \sqrt{D}}{2}\right).$$

From left to right, the columns of the table are N , the absolute values of the integers $\Theta_{[R]}$ for each $[R] \in \mathcal{R}$, the number, denoted $\#\Theta_{[R]}$, of classes $[Q] \in Cl(-N)$ with value $\pm\Theta_{[R]}$ (this equals $h_R(-N)$ by Theorem 3.3), and the values $h_{[\mathfrak{a}, R]}^\varepsilon(-N)$.

For $D = -7$, the type number is 1 and so $\#\Theta_{[R]} = \frac{1}{2}h_R(-N) = h(-N)$ gives the N -th coefficient of the weight $3/2$ level $4D$ form $\frac{1}{2} + \omega_R \sum_{N>0} H_D(N)q^N$ defined by the modified Hurwitz invariants $H_D(N)$ (see [Gro84, p. 120] for their definition).

N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h_{[\mathfrak{a}, R]}^\varepsilon(-N)$	N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h_{[\mathfrak{a}, R]}^\varepsilon(-N)$
11	1	1	-1	107	1	3	-3
23	1	3	-1	127	1	5	1
43	1	1	1	151	1	7	-1
67	1	1	-1	163	1	1	1
71	1	7	-3	179	1	5	-3
79	1	5	-1	191	1	13	-5

TABLE 1. $D = -7$, $N \leq 200$, $t = 1$.

N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h_{[a,R]}^\varepsilon(-N)$	N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h_{[a,R]}^\varepsilon(-N)$
23	0	2	2	103	0	3	3
	2	1	1		2	2	2
31	0	2	2	163	0	1	1
	2	1	-1		2	0	0
47	0	3	3	179	0	2	2
	2	2	2		2	3	1
59	0	2	2	191	0	8	8
	2	1	-1		2	5	1
67	0	0	0	199	0	5	5
	2	1	-1		2	4	4
71	0	4	4	223	0	4	4
	2	3	-3		2	3	3

TABLE 2. $D = -11$, $N \leq 250$, $t = 2$.

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